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NOTE

ON THE MEMBERSHIP PROBLEM FOR THE ELEMENTARY CLOSURE OF A POLYHEDRON

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The problem: Given an integral matrix A, an integral vector b and some rational vector $\hat{x}$, decide whether $\hat{x}$ is outside the elementary closure $\{x\in\mathbb{R}^n \mid Ax\le b\}'$, is NP-complete. This result is achieved by an extension of a result by Caprara and Fischetti.

1. Introduction

Let $P\subseteq\mathbb{R}^n$ be a polyhedron. A *Gomory–Chvátal cut* (G-C cut for short) of P is an inequality of the form $c^Tx\le\lfloor\delta\rfloor$, where $c\in\mathbb{Z}^n$ is an integral vector and $c^Tx\le\delta$ is valid for P. The *elementary closure* P' of P is the intersection of P with all G-C cuts of P. The following result can be found in [2, proof of Lemma 6.34].

Lemma 1. *Let P be the polyhedron $P=\{x\in\mathbb{R}^n \mid Ax\le b\}$ with $A\in\mathbb{Z}^{m\times n}$ and $b\in\mathbb{Z}^m$. The elementary closure P' is the polyhedron defined by $Ax\le b$ and the set of all inequalities $\lambda^TAx\le\lfloor\lambda^Tb\rfloor$, where $\lambda\in[0,1)^m$ and $\lambda^TA\in\mathbb{Z}^n$.*

Consider the following decision problem.

Definition 1. (MEC) The *membership problem for the elementary closure* is as follows:

Given an integral matrix $A\in\mathbb{Z}^{m\times n}$, an integral vector $b\in\mathbb{Z}^m$ and a rational vector $\hat{x}\in\mathbb{Q}^n$, is $\hat{x}\notin\{x\in\mathbb{R}^n \mid Ax\le b\}'$?

It is easy to see that MEC is in NP. In fact, if $\hat{x}$ is not in the elementary closure $\{x\in\mathbb{R}^n \mid Ax\le b\}'$, then there exists a G-C cut $c^Tx\le\lfloor\delta\rfloor$, which is not

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satisfied by $\hat{x}$ such that c can be written as $c^T = \lambda^T A$, where $\lambda \in [0,1]^m$. Notice then that $\|c\|_\infty \leq \|A^T\|_\infty$. Clearly $\hat{x}$ does not satisfy the inequality $c^T x \leq \lfloor \gamma \rfloor$, where $\gamma = \max\{c^T x \mid Ax \leq b\}$. Since linear programming is polynomial, this c serves as a polynomial certificate for the fact that $\hat{x}$ is not in $\{x \in \mathbb{R}^n \mid Ax \leq b\}'$.

Here we show that MEC is NP-complete. This solves a problem raised by Schrijver in [5, p. 351].

Let $A \in \mathbb{Z}^{m \times n}$ be an integral matrix, $b \in \mathbb{Z}^m$ be an integral vector, and let $P \subseteq \mathbb{R}^n$ be the polyhedron $\{x \in \mathbb{R}^n \mid Ax \leq b\}$. A $\{0, \frac{1}{2}\}$-*cut* derived from A and b is a G-C cut of P of the form $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, where $\lambda^T A$ is integral and the components of λ are either 0 or $\frac{1}{2}$. The $\{0, \frac{1}{2}\}$-*closure* $P_{\frac{1}{2}}(A,b)$ derived from A and b is the intersection of P with all the $\{0, \frac{1}{2}\}$-cuts derived from A and b.

Definition 2. (M0$\frac{1}{2}$) The *membership problem for the* $\{0, \frac{1}{2}\}$-*closure* is as follows:

> Given an integral matrix $A \in \mathbb{Z}^{m \times n}$, an integral vector $b \in \mathbb{Z}^m$ and a rational vector $\hat{x} \in \mathbb{Q}^n$, is $\hat{x} \notin P_{\frac{1}{2}}(A,b)$?

Caprara and Fischetti show in [1] that M0$\frac{1}{2}$ is NP-complete. For the sake of completeness we state and prove their result below.

2. M0$\frac{1}{2}$ is NP-complete

This section follows closely [1, Sect. 3]. Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ be integral and let $\hat{x} \in \{x \in \mathbb{R}^n \mid Ax \leq b\}$. The vector $\hat{x}$ does not satisfy all $\{0, \frac{1}{2}\}$-cuts derived from A and b if and only if there exists some $\mu \in \{0,1\}^m$ with $\mu^T A \equiv 0 \pmod 2$ and $\mu^T b \equiv 1 \pmod 2$ such that the inequality $\mu^T (b - A\hat{x}) < 1$ is valid.

Definition 3. (WBC) The *weighted binary clutter problem* is the following:

> Given a matrix $Q \in \{0,1\}^{r \times t}$, a vector $d \in \{0,1\}^r$ and a weight vector $w \in \mathbb{Q}^t_{\geq 0}$, decide whether there exists some $z \in \{0,1\}^t$ with
>
> $$Qz \equiv d \pmod 2 \text{ and } w^T z < 1.$$

WBC is NP-complete, since for example the problem of decoding of linear codes [3, p. 280] can be reduced to it.

We will see that one can reduce WBC to both M0$\frac{1}{2}$ and MEC, which implies that they are both NP-complete.

Theorem 2. (Caprara and Fischetti) M0$\frac{1}{2}$ *is* NP-*complete.*

Proof. M0$\frac{1}{2}$ clearly is in NP. We show that WBC can be polynomially reduced to M0$\frac{1}{2}$.

For this let Q, d and w be an instance of WBC. Construct the following instance of M0$\frac{1}{2}$:

$$A = \left(\begin{smallmatrix} Q^T \\ d^T \end{smallmatrix} \middle| \; 2I_{t+1} \right)$$
$$b = (2, \dots, 2, 1)^T$$
$$\hat{x} = (\mathbf{0}^T, \mathbf{1}^T - \tfrac{1}{2}w^T, \tfrac{1}{2})^T,$$

where $\mathbf{0} = \{0\}^r$ and $\mathbf{1} = \{1\}^t$. Notice that $b - A\hat{x} = (w_1, \dots, w_t, 0)^T$. The point $\hat{x}$ does not satisfy all $\{0, \frac{1}{2}\}$-cuts derived from A and b if and only if there is a $\mu \in \{0,1\}^{t+1}$ with $\mu^T A \equiv 0 \pmod 2$, $\mu^T b \equiv 1 \pmod 2$ and $(w_1, \dots, w_t, 0)\mu < 1$. This is the case if and only if there is a $z \in \{0,1\}^t$ with $Qz \equiv d \pmod 2$ and $w^T z < 1$. ∎

3. MEC is NP-complete

Lemma 3. *Let P be the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ with A and b integral. If A is of the form $A = (C \mid 2I_m)$ for some integral matrix C, then $P' = P_{\frac{1}{2}}(A, b)$.*

Proof. Clearly $P_{\frac{1}{2}}(A,b) \supseteq P'$. For the reverse inclusion we simply show that each undominated G-C cut of P is also a $\{0, \frac{1}{2}\}$-cut derived from the system (A, b). Recall that each undominated G-C cut of P can be written as $\lambda^T A x \le \lfloor \lambda^T b \rfloor$, where $\lambda^T A \in \mathbb{Z}^n$ and $\lambda \in [0,1)^m$. However λ has to satisfy $\lambda^T 2I_m \in \mathbb{Z}^m$. Thus for $i = 1, \dots, m$ one has $2\lambda_i \in \mathbb{Z}$ and $0 \le 2\lambda_i < 2$, i.e., $\lambda \in \{0, \frac{1}{2}\}^m$. ∎

Corollary 4. MEC *is* NP*-complete.*

Proof. We reduce WBC to MEC. Let Q, d and w be an instance of WBC. Construct an instance of MEC as given in the proof of Theorem 2. Since in this case $P_{\frac{1}{2}}(A,b) = P'$ the claim follows. ∎

Theorem 5. *If* P $\neq$ NP, *then optimizing over the elementary closure of a rational polyhedron cannot be done in polynomial time.*

Proof. If one could optimize over the elementary closure of a rational polyhedron in polynomial time, then one could also solve the separation problem for the elementary closure in polynomial time (see [4]), which is at least as hard as MEC. ∎

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